# LEARNING MATHEMATICS FROM MULTIPLE REPRESENTATIONS: TWO DESIGN PRINCIPLES

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*This paper describes two design principles for designing mathematics tasks using technology. These are:* The parallel instantiations principle. *Presenting students with a large number of non-prototypical instantiations simultaneously and non-transiently perturbs their thinking and supports thinking-in-change.* The discriminating tools principle: *Discriminating tools enable children to differentiate between the tools' feedback to acquire knowledge from their use. Students will learn within the task rather than merely from the task. These principles were first developed in a study on 9-11 year olds' geometric defining and later applied and amended in a larger study that encouraged 9-11 year olds' conceptual understanding of fractions. The paper presents how the principles were applied within the two studies.*

*Design-based research, design principles, task design, exploratory learning environment, multiple representations*

## INTRODUCTION

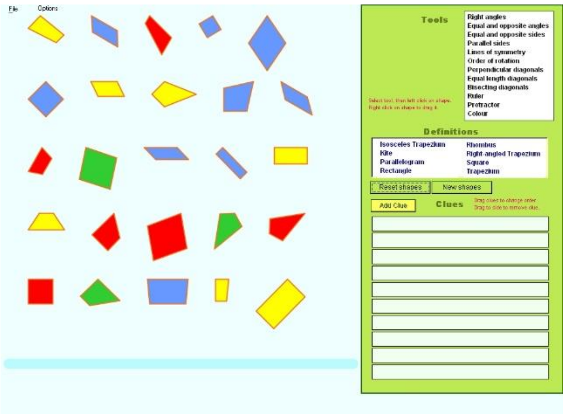
Despite the plethora of technology-based applets, games and websites available for students and teachers, there remains a significant proportion that are developed by those with technological understanding but not necessarily with knowledge of mathematics education. We can generally consider two ends of the spectrum. At one end, perhaps due to lack of imagination or due to lack of resources, there are the 'low hanging' fruits of simple flashcard-like arithmetic, matching games, drag-and-drop onto number lines or simply inputing a number and receiving basic feedback. At the other end those with often very complex and difficult-to-access or use in class. We see a gap in those more interactive pieces of software that involve virtual manipulatives, perhaps because it may not be clear how to design these. One of the aims behind the design-based research that we have been undertaking is to seek, in the context of different projects, principles for designing this type of mathematics software.

In this paper we discuss two principles that originally evolved from a PhD study (Hansen, 2008) focusing on the geometric defining of 9-11 year olds. The principles were later implemented in a larger study (www.iTalk2Learn.eu) that aims to develop an open-source intelligent tutoring platform that supports fractions learning for students aged 5 to 11. We offer these design principles as a contribution to the design-based research community. We reflect upon how the principles worked within two studies. The paper is outlined as follows. The remainder of the introduction provides the wider methodological context of both studies, design-based research, outlining how design principles are a common outcome from this process. The second and third sections introduce *Quads* and *Fractions Lab*, the environments from each study utilising virtual manipulatives, and a selection of the tasks students completed within them. The principles themselves are introduced in the fourth and fifth sections. Within each section a justification from the literature related to the principles and data or findings are shared. The conclusion brings the two principles together, demonstrating their symbiotic relationship.

Educational design research involves the development of a tangible outcome that could be an educational product, process, programme (of CPD) or policy (McKenney & Reeves, 2014) using interventionist, iterative, process-orientated, utility-orientated and theory-orientated methods (van den Akker, Gravemeijer, McKenney & Nieveen, 2007). Because design research is situated within numerous domains, methods vary, and the outcomes are unique and often descriptive because the designer determines the next steps from what the specific context dictates (Visscher-Voerman & Plomp, 1996). However, design experiments typically include a phase that produce trustworthy resulting claims (Cobb et al, 2003:12), some of which can be presented as design principles (McKenney, Nieveen & van den Akker, 2006; Wang & Hannafin, 2005). Such principles are situated between "scientific findings, which must be generalized and replicable, and local experiences or examples that come up in practice" (Bell, Hoadley, and Linn, 2004:83) and are often presented as heuristic guidelines (van den Akker, 1999) which are intended to "help others select and apply the most appropriate substantive and procedural knowledge for specific design and development tasks in their own settings" (McKenney, Nieveen & van den Akker, 2006). As design principles are refined by others adapting them to their own experiences (Bell, Hoadley, and Linn, 2004), they become more fine-tuned.

Here we present two principles for designing environments using virtual manipulatives and their associated tasks. The principles were identified after observations of how students were interacting with *Quads* while following three different tasks. They later acted as guiding criteria for design decisions and were fine-tuned when designing *Fractions Lab*. These design principles have their roots in cognitive load and instructional multimedia aids theories and a selection of the literature informing the original design decisions that led to these principles is discussed here. *Quads* and its three games, and *Fractions Lab* with its related tasks are discussed below; the principles follow.

## QUADS



Select tool, then left click on shape. Right click on shape to drag it.

Drag clues to change order. Drag to side to remove clue.

**Fig. 1: Quads (Hansen, 2008)**

After three previous design iterations, Hansen (2008) designed *Quads*, a virtual environment inspired by the popular board game, "Guess Who?" (Milton Bradley Company, 1987) that enables students to play three geometry-definition games involving quadrilaterals. Essentially, 25 instantiations (referred to as *mightbes*) are positioned on the screen (see Figure 1). In pairs, students (the "clue-setters") select one figure/definition and generate property-related clues for a second pair (the "clue-followers") to follow in order to identify the chosen figure/definition. Clues are generated using *musthaves*. These are inclusive statements that refer to the properties that particular figures 'must have' to belong to a definition. A further game explores notions of necessary and sufficient properties. *Quads* was trialled by 32 students aged 9-11 years from four schools.

## FRACTIONS LAB

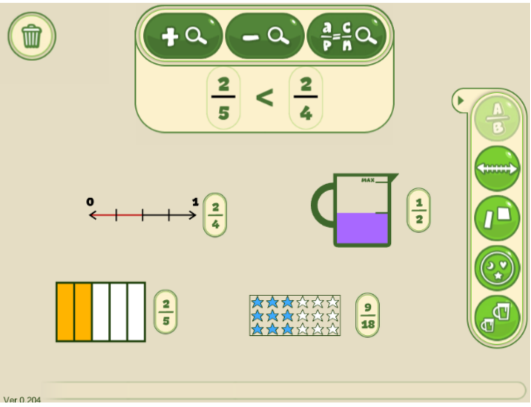


Figure 2: Fractions Lab (www.iTalk2Learn.eu)

*Fractions Lab* is an exploratory learning environment that acts as a stand-alone program or as a component of the iTalk2Learn project's (www.italk2learn.eu) intelligent tutoring system. Students are given tasks that require them to construct models from a range of representations (see Figure 2) and act upon them to challenge common fraction misconceptions (Hansen, 2014). Tools enable a student to change the numerator and denominator, partition the models, change the colour or copy a fraction. The addition, subtraction and comparison tools (at the top of the screen in Figure 2) allow students to check their hypotheses. The iteration of *Fractions Lab* used in this discussion involved 32 students aged 9-10 and 37 students aged 10-11 years from one school.

## THE PARALLEL INSTANTIATIONS PRINCIPLE

*When presenting students with non-prototypical instantiations simultaneously and non-transiently, children's thinking is perturbed and this supports thinking-in-change.*

When students create their own representations of a concept, it is limited to their own understanding and often reflects misconceptions. However, presenting representations to students can lead to inappropriate interpretations (Handscomb, 2005). There is a wide range of factors that influence learning using multiple representations, including the number of representations that are presented, either at any one time or at some point during the session (Ainsworth, 2006), as well as the mode of presentation. Multi-modal representations support the dual processing in the working memory (Mayer, 1999; Sweller, 1999). In our work we have used static and dynamic representations with symbolic text because "pictures have more features available for processing than do words, and pictures may help access meaning more quickly and completely than words … text conditions also allow the learner to process the verbal information at the learner's pace" (Najjar, 1998:312).

### The parallel instantiations principle in Quads

Great care was taken to design 25 instantiations (*mightbes*) (see Figure 1) ensuring a range of *musthaves* within each definition and a variety of prototypical and non-prototypical representations that can be refreshed. Each time the game starts over, or the 'new shapes' button is pressed, the instantiations change within programmed constraints. The instantiations were designed to challenge students' 1) visual perceptions of individual instantiations and 2) understanding of what figures constitute a definition. For example a figure may appear as if it has right angles, but on closer inspection with the protractor tool it has none. In relation to 2), the highlighting function shows the figures that sit within a particular definition.

Although an "excessive number of representations rarely helps learning" (Ainsworth, 2006) the number of instantiations (25) was settled upon through consideration of the range of definitions and *musthaves* to be covered and the physical space on the screen, as well as needing to provide counter-examples (Fischbein, 1987).

The highlighting function was used by students to identify the instantiations that exemplified a given definition. This often challenged students' pre-conceptions. For example, in every instance that a pair of students (n=5) selected the kite definition they were surprised that the set included instantiations of squares and they made comments such as "That doesn't look like a kite, that's a square" or that they "don't look right." Many of the other children tended to initially express surprise or admitted they were challenged by the figures presented to them within the highlighted sets. This was seen within a number of definitions, for example squares in the rhombus set (2 pairs), squares in the right angled trapezium set (3 pairs), squares in the isosceles trapezium set (1 pair), squares in the trapezium set (1 pair). However, by the end of the task students were able to define each set and competently explain why all the instantiations sat legitimately within the definition (Hansen, 2008).

### The parallel instantiations principle in Fractions Lab

In *Fractions Lab* it is possible to present students with tasks that utilise four different graphical models (area, number line, sets and liquid measures), each with its associated fraction symbol (see Figure 2). One task provides four fractions and asks students to find the odd one out. Another task asks students to make a fraction using each of the models. In tasks where students are asked to manipulate existing fraction models, they are able to do so while leaving the original intact, something that is not easily achievable using physical manipulatives (Olive & Lobato, 2007).

We have emerging evidence that parallel instantiations have an impact on students' understanding of how fractions may be represented (Hansen, Geraniou & Mavrikis, 2014). For example, when we asked 35 10-11 year old students to draw as many ways as possible to show ¼ (before using Factions Lab), none drew a number line or in a jug. Yet, after using *Fractions Lab* for a relatively short time (10-15 minutes), 89% of students later added a measuring jug and 77% a number line. Furthermore, in an open-ended question about what they had learnt when using *Fractions Lab*, 18 of the 35 students stated that they learnt more about the way fractions were represented. Their written comments included statements such as, "Fractions can be presented in many ways", "You have a variety of choices to represent fractions", and "You can show fractions with liquid". This is worthy of note when most instruction materials and teachers, rely on the limited part/whole representation (Alajmi, 2012; Baturo, 2004; Pantziara & Philippou, 2012) and our personal experience suggests that students would not provide such reflective statements around multiple fraction representations unprompted.

## THE DISCRIMINATING TOOLS PRINCIPLE

*By providing tools to carry out the tasks that require pre-requisite knowledge to achieve the objective (but if undertaken manually would detract from it) students will learn within the task rather than merely from the task. Discriminating tools enable children to differentiate between the tools' feedback to acquire knowledge from their use.*

Novices (in this case, students) often fall back on weak problem-solving strategies because they do not possess the schemata to support the work they are undertaking (Sweller, 1999). This is an issue when designing for challenging or complex mathematical concepts such as geometric defining and fractions because of the multiple facets to their nature. In light of this Hansen (2008) developed tools to free up the working memory to focus on achieving the objective of the task (Kalyuga, Chandler & Sweller, 1999) and provide procedural information that is prerequisite for learning to take place in a complex task (Van Merrienboer & Kirschner, 2007). Students consider feedback from tools, discriminating between instantiations and their built-in epistemic constraints. In doing so the tools are catalyst for learning within the task. For example, a protractor tool providing a figure's interior angles enables students to focus on their line of thought regarding the property of a 'number of equal and opposite angles' while they investigate if a figure contains them without having to manually carry out measuring procedures. In *Fractions Lab* a 'find equivalent' tool showing how a fraction symbol changes while a model is partitioned enables students to think about the relationships within and between equivalent fractions rather than carrying out rote multiplication procedures on the numerator and denominator.

## The discriminating tools principle in Quads

The discriminating tools in *Quads* requires students to be attentive to the properties of instantiations. A tool can be selected from a menu on the screen (see Figure 3) and data about figures are displayed in static form (e.g. the internal angles are given for the protractor) or dynamic form (e.g. order of rotation, see Figure 4).

Select tool, then left click on shape.
Right click on shape to drag it.

**Figure 3: Quads discriminating tools**

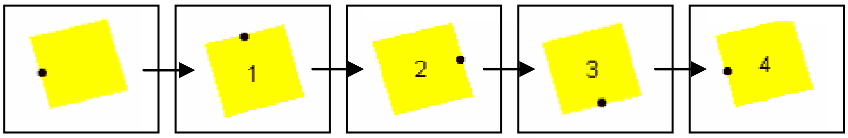


**Figure 4: Five screen shots from the Order of Rotation animation showing a square with Order 4**

When clue-setters or clue-followers, students were able to efficiently work through the properties of the *mightbes* to make clues or to identify the shapes their partners had selected. In an earlier iteration with a physical manipulatives task, Hansen (2008) noted that students did not engage with properties to the same depth, often using incorrect visual cues. Using the tools, the students worked through many more instantiations than they could have using a real protractor or ruler and at the same time.

## The discriminating tools principle in Fractions Lab

*Fractions Lab* uses tools to manipulate fraction representations by partitioning to make equivalents), adding and subtracting (see Figure 2, top of screen). Students are able to make equivalent fractions using the partition tool. As the models are manipulated, their corresponding symbols change. The addition and subtraction tools use animation to show the results of fractions being joined or taken away. These three tools provide feedback to the students to check their activity.

When asking 35 10-11 year old students an open-ended question about what they had learnt after using *Fractions Lab* for a short duration (10-15 mins), 60% referred to an aspect related to the discriminating tools (15 referred to addition or subtraction and six to equivalence, others referred to the size of fractions or representations). For example, "Before you add two fractions together you need to make sure that both denominators are the same", "It has made me more confident at adding and subtracting fractions" and "You can double the numerator and denominator and it equals the same."

The equivalence tool is discussed in detail in Hansen, Mavrikis, Holmes & Geraniou (submitted). Here we briefly discuss the addition tool. In *Fractions Lab,* if a student attempts to add two fractions with unlike denominators, *Fractions Lab* will refrain from providing the answer. It is only when the denominators are the same that an answer will be given. In this case, when students attempt to add two fractions with unlike denominators the feedback they receive typically does not give students their expected or desired outcome (i.e. the answer). As a result, the system encourages them to use the equivalence tool to make fractions with like denominators before adding them together. The feedback from the addition tool appears to have supported students to learn within the task rather than merely from it and this situated abstraction is a stepping stone to adding fractions with unlike denominators procedurally with understanding.

21 9-10 year old students who had not been introduced to addition and subtraction of fractions with unlike denominators were asked to consider the work of a fictitious student (2/3 + 1/6 = 3/9) while using *Fractions Lab*. When asked to provide an explanation for why they thought the student was correct or incorrect, 19 out of the 21 stated the student was incorrect and gave a plausible justification (see Table 1).

| | |
|---|---|
| Comparing the size of various fractions, e.g. "2/3 and 1/6 put together are bigger than 3/9 and 2/3 is bigger than 3/9 to start with." | 2 |
| Identification of fictitious students' misconception, e.g. "he has added the numbers together." | 4 |
| Refers to need to change denominator / denominators need to be the same, e.g. "I tried it on *Fractions Lab* so the denominator needs to be the same." | 5 |
| Refers to partitioning / equivalence, e.g. "because if you partition 2/3 it will go to 4/6 which will make it easier." | 8 |

**Table 1. Types of response given**

Of the two who did not provide a clear, correct explanation one student wrote, "It looks right but it isn't" and the other wrote "I think it is because you don't add up the denominator. I am not so sure though."

## CONCLUSION

The presentation of multiple parallel instantiations and discriminating tools that enable students to act upon the instantiations are very difficult or even impossible to re-enact with physical manipulatives. We, therefore, offer these two design principles as guiding criteria for designing affordances that have the potential to support students' conceptual understanding in mathematics.

There is a symbiotic relationship between the two principles we have presented. Students' thinking is initially perturbed by being presented with non-prototypical instantiations, yet feedback provided by the discriminating tools enables students to acquire knowledge within the activities rather than from them. For example, in *Quads* the non-prototypical instantiations forced students to use the discriminating tools to check properties more than earlier iterations of Hansen's (2008) work had elicited. In *Fractions Lab* the constraints built into the addition tool encouraged students to find equivalent fractions as a step towards adding fractions with unlike denominators successfully. We claim therefore that the students used the feedback to acquire knowledge from the use of the discriminating tools.

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