DESIGNING INTERACTIVE REPRESENTATIONS FOR LEARNING FRACTION EQUIVALENCE

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This paper describes a study that investigated the use of a tool in an exploratory learning environment (ELE) designed to support students' understanding of equivalent fractions. The study, part of a larger project, involved 67 9-11 year old students in England. It addressed the question: How does a partitioning tool support students' conceptual understanding of equivalence? Data were collected through observations, students' written work in an equivalence task, and a written self-reflection on learning at the end of their time in the ELE. Results showed that using the partitioning tool with an area representation was instrumental in challenging some students' preconceived ideas about equivalent fractions and that the students were able to develop situated abstractions about fraction equivalence.

Keywords: equivalent fractions, exploratory learning environments, design, partitioning, fraction representations

INTRODUCTION

It is widely accepted that fractions are a complex and difficult aspect of mathematics education. An early emphasis on whole-number constructs using additive structures foregrounds the multiplicative structures used with rational numbers, while this whole-number bias (Behr, Harel, Post & Lesh, 1994; Schmittau, 2003) appears to be a barrier within many fraction equivalence misconceptions. As a result, students often inappropriately apply whole-number schemes, instead of seeing a fraction as a quantity or quotient relation between two numbers, and default to seeing it as two separate numbers (Murray & Newstead, 1998). Furthermore, fraction tasks (especially computerised ones) are often limited in scope and their approach is typically instructional and procedural.

Using a Design-Based Research methodology (Design-Based Research Collective 2003), we have developed an exploratory learning environment, Fractions Lab, that allows students to interact with various fractions representations, add or subtract them, and check their equivalence. In this paper, we explore how our design decisions related to fraction equivalence tasks enabled students to build upon their intuitive thinking about fractions (Mack, 1990) and challenged them to reflect on the feedback they received.

FRACTIONS LAB

Fractions Lab is an exploratory learning environment that acts as a stand-alone program or as a component of the iTalk2Learn project's (www.italk2learn.eu) intelligent tutoring system. It aims at fostering conceptual knowledge, which we define as implicit or explicit understanding about underlying principles and structures of a domain (Rittle-Johnson & Alibali, 1999). The focus of this type of knowledge lies on understanding why, for example, different mathematical principles refer to each other and on making sense of these connections. Conceptual understanding of equivalent fractions, for example, includes students being able to make connections between fraction representations by understanding what is the same and different within them (Lesh et al., 1983) and showing that a fraction represents a number with many names (Wong & Evans, 2007).

Fractions Lab adopts a holistic approach, encourages student-directed activity, adopts a constructivist stance to learning and assumes an active role for the student (Ben-Naim, Marcus, & Bain, 2008). Tasks that would support students' conceptual development within the Fractions Lab environment were also developed to support students to address common misconceptions and conceptual understanding. Students are given tasks that, for example, ask them to (i) construct three fractions equivalent to 3/5, (ii) find the odd fraction out, (iii) make a fraction equivalent to ¾ that has a denominator of 12, and (iv) explain whether another student is correct or incorrect when they state "2/6 = 1/12 because 2x6 = one 12".

FRACTIONS LAB DESIGN DECISIONS

We took several design decisions that involve the use of virtual manipulatives as fraction representations (area, number line, sets of objects, liquid measures and symbols) (Lamon, 2012) and complementary tools. In this paper we discuss three graphical representations (rectangle, number line, measuring jug, because sets were unavailable in this iteration) and the partitioning tool, a tool designed to support students' understanding of equivalent fractions.

In Fractions Lab, students construct their own fractions using the virtual manipulatives within the learning environment. Typically, virtual manipulatives include additional features and options that develop what a physical manipulative offers and they can also represent situations that are not even possible with physical manipulatives (Moyer, Bolyard & Spikell, 2002, Reimer & Moyer, 2005, Steen et al, 2006; Suh & Heo, 2005). When using teacher-led virtual manipulatives involving fraction equivalence, Suh & Heo (2005) found students linked graphical and symbolic representations, experimented and tested hypotheses (using trial and error in a non-threatening environment), spent longer on task, and appeared to model the fluid nature of their thinking.

We designed Fractions Lab to enable students to perform actions on representations that they would not be able to with physical manipulatives. For example, it is possible to establish a relation between a part and a whole by partitioning a rectangle, changing the denominator and numerator while leaving the original whole intact, something that was not previously possible without destroying the original (Olive & Lobato, 2007).

Tools

Students can manipulate each of the representations by using various tools to partition, add or subtract fractions, while executing tasks that challenge common fraction errors and misconceptions. We designed the tools to enable a student to offload what would be otherwise a heavy and error-prone cognitive burden (Pea, 1993). Additionally, students are able to undertake "profoundly different" (Nardi, 1998) actions that "enable us to act, perceive and reason beyond our natural limits" (Nunes, 1997:30). Clarebout, Elen, Johnson & Shaw (2002) suggest that support tools should be embedded into a software task and that students should be able to choose to what extent they use them. However, students must enter into a relationship with the tools to "afford the user expressive power: the user must be capable of expressing thoughts and feelings with it. It is not enough for the tool to merely 'be there', it must enter into the user's thoughts, actions and language" (Noss and Hoyles, 1996:59). In this paper we focus on just one of the tools available in Fractions Lab: the partitioning tool.

The partitioning tool

The partitioning tool was designed to support students' understanding of equivalence. Figs. 1, 2 and 3 show the stages of three different representations being partitioned. It was our aspiration that students would use the partitioning tool to split up the parts of the representation already shown to reflect the original fraction made and, with carefully-designed tasks, notice patterns within the changing fraction symbol.

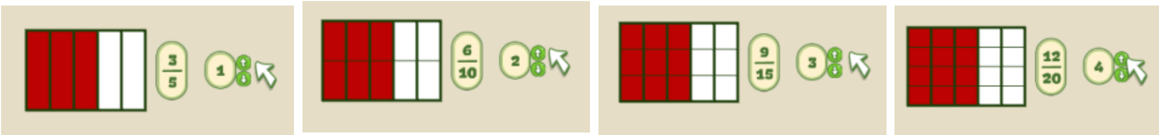

Figure 1: 3/5 in a rectangle, partitioned three times. The fraction symbol changes to reflect the new fraction name.

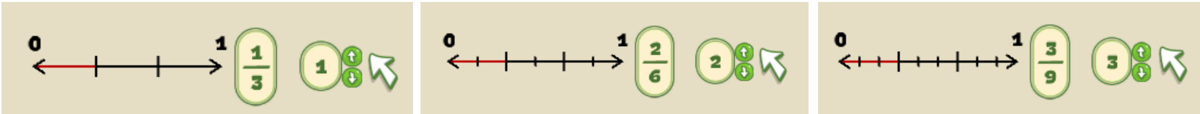

Figure 2: 1/3 on a number line, partitioned twice. The fraction symbol changes to reflect the new fraction name.

Figure 3: 3/4 in a measuring jug, partitioned three times. The fraction symbol changes to reflect the new fraction name.

METHOD

We validated our design decisions based on an analysis of a series of 'design experiments' (Cobb *et al.*, 2003) that provide support for the efficacy of Fractions Lab and the lessons learned for interactive fraction representations. In the iteration reported in this paper, three representations were available to the students. These representations were area, number line and liquid measures.

We worked with 32 Year 5 (9-10 year old) and 35 Year 6 (10-11 year old) students from one school near the end of their academic year, visiting each cohort once over a period of two days in June and July. All the participants and their parents gave informed consent and permission for the study was obtained from the school's staff. The purpose of the study was to evaluate the impact of the tools and the effectiveness of the tasks on students' conceptual understanding of fractions equivalence, addition and subtraction. Prior to using Fractions Lab the students had not met the notion of partitioning a fraction to find an equivalent.

During each visit, each student worked with Fractions Lab for a duration of 15 - 30 minutes, undertaking a small and varied selection of tasks, such as those outlined above. 18 Year 5 students completed a task to familiarise them with the partitioning tool. All students (n=67) completed a 'reflection on my learning' questionnaire afterwards. We captured all students' work on the screen and recorded their speech. We periodically intervened to gain further insights into the students' thinking-in-change or to aid their reflection on the task.

FINDINGS

First, we present a case study of a student completing one equivalence fraction task in the ELE, to demonstrate their thinking-in-change about fractions. We follow the case study with data from the partitioning tool task and the reflection questionnaire, demonstrating the impact Fractions Lab had more generally on some of the students' thinking about equivalent fractions.

1. Case Study

We first present a case study of a student completing the task *Explain whether another student is correct or incorrect when they state "2/6 = 1/12 because 2x6 = one 12"*. Initially, the student (who we will call George) believes the statement is correct. The case study demonstrates how, over 12 minutes 45 seconds, and with feedback from Fractions Lab, George's thinking changes.

1	George:	2/6 = 1/12 because two sixes equals twelve. [Constructs 2/6 and 1/12 and uses the 'compare tool' to check]. Oh. This [computer] is wrong.
2	Researcher:	Why do you think it is wrong?
3	George:	The computer is wrong because I know that 1/12 equals 2/6. Look, um, 2/4 = 1/8.
4	George:	I know one twelfth equals two sixths [pause] or is it I'm wrong? I don't know ... I don't know which one it is. [Period of trial and error. The student finds that 2/4 and 4/8 are equivalent].
5	George:	Yay! It worked!
6	Researcher:	Why do you think it worked?
7	George:	I know what I did wrong ... need to have one twelfth equals twelve [pause] fourths, no, no, no I'm confused. [Continues to explore why 2/4 and 4/8 are equivalent].
8	George:	So what makes this one work that the other one doesn't? [pause] [points to denominators] Two 4s equals four 8s. What's four 8s? 32. So I need to make a thirty two[th].
9	Researcher:	Why don't you try out 2/6 and 1/12 to find two fractions that are equivalent?

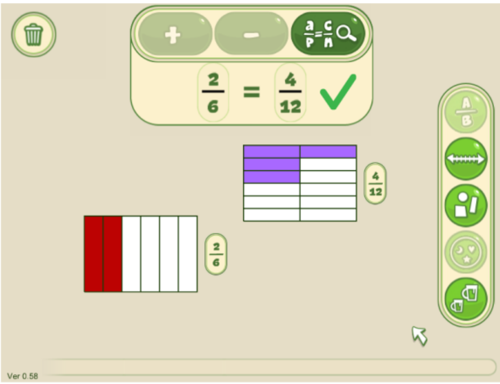

Figure 4: Screenshot of George's work at line 10

10	George:	[Makes 2/6, then partitions 1/12 into 2/24. He changes the denominator to 26 and the numerator to 4. He changes the number of partitions until he makes a denominator of 10]. Wait. [Makes a denominator of 12 then 14]. Wait, is it 12? Let's try that one. [Makes a denominator of 12 (so the fraction is 4/12). Checks the two fractions are equivalent]. Yay!
11	Researcher:	Why do you think that is correct?
12	George:	What is four twelves? Forty eight. [Pause] I know what I did. It's got to make up the same number. So they're a bit the same. They are in the same times table. One's 4 and one's 2, so two 2's equals 4 and then two 6's is 12.

George uses the rectangle representation throughout, which is representative of other students' working. We suspect that this is due to students being most familiar with area representations (Duval, 2006), but it may also be because of the central position of the area button in Fractions Lab. George held the misconception "x/y = 1/xy" strongly. Initially he believed the computer was wrong (line 1) and tested his conjecture using 2/4 = 1/8 (line 3). Through trial and error he found that 2/4 and 4/8 were equivalent (line 4) and his thinking is perturbed (lines 7-8). He begins to make a link between the numerators and denominators of the two equivalent fractions (line 8). After using the partitioning tool he tests 4/12 to see if it is equivalent to 3/4 and finds that it is. He demonstrates intuitive multiplicative reasoning by stating that the numerator of each equivalent fraction and denominator are "in the same times table", and we can see that he appears to be overcoming his use of additive structures and whole-number bias by stating the relationship between the two fractions using multiplication.

2. Partitioning tool task

18 Year 5 students completed a familiarisation task using the Partitioning Tool. They were required to make a fraction, note which representation they used, write the resulting fractions of three partitions and explain in their own words, for another child, how partitioning works. 16 students used the area representation, the number line and liquid measures representations were each used by one student. Table 1 shows the types of students' responses.

Explanations related to...		
multiples or multiplication (e.g. it times by the number you partition by)	7	39%
doubling (e.g. double the numerator and denominator)	4	22%
splitting the number (e.g. partitioning works by the number being split up)	2	11%
addition (e.g. on the top number you add 5 and on the bottom number you add 10)	2	11%
splitting / cutting up the rectangle to make smaller pieces (e.g. cutting it up into smaller pieces)	3	17%
Total:	**18**	**100%**

Table 1. Partitioning Tool task: Students' explanations of how the tool worked

No students referred to a procedural method (e.g. "times the top and bottom by the same number") but instead focused on the action of the partitioning tool on the representation. 15 (73%) of the 18 students commented on the change on the symbol as a result of the action. The remaining students (17%) focused on the representation (rectangle) itself.

3. Written reflections

All the students (n=67) were asked to reflect on their learning using Fractions Lab. There was no clear difference between the responses of the Year 5 and Year 6 so the data are combined. Because the questions allowed for open-ended responses, many students focused on aspects beyond the scope of this paper, such as fraction size, addition or subtraction.

All students were asked how each of the representations helped them to understand fractions. 17 of the 67 students made statements related to equivalence. These are presented in Table 2. When considering *how each of the representations helped students to learn*, a quarter referred to equivalence. More students referred to rectangles supporting their learning of equivalent fractions compared to the number line and liquid measures. This may be because not all the students will have undertaken equivalence tasks with those representations.

Indicative student statements	Totals:
General statements about finding equivalence, e.g. • The equivalent. • It helps me actually visualise the fractions as decimal numbers. • How to show equivalent fractions. • How to find equivalent fractions.	6
Visual comparison, e.g. • Which fractions are the same because it colours in the rectangle. • Equivalent fractions. Put a fraction above another to see if they were the exact same. • How fractions look the same but have different numbers 1/2 = 2/4.	5
Partitioning, e.g. • They could be partitioned. • How partition works. • To partition instead of times by two.	6
	17

Table 2. Student responses related to equivalence

DISCUSSION

The main objective of our work is to challenge students' pre-conceived ideas of how fractions are represented and how Fractions Lab can create an environment for students to develop situated abstractions about equivalent fractions. Prior to using Fractions Lab the students had not met the notion of partitioning a fraction to find an equivalent.

The case study provides a window into one student's thinking-in-change, as he challenges his own misconception about equivalent fractions. George appears to have entered into a relationship with the representations and tools in Fractions Lab which gave him expressive power, entering his thoughts, actions and language (Noss and Hoyles, 1996). It also shows how George's whole-number bias (Behr et al, 1983; Newstead & Murray, 1998; Schmittau, 2003) remained dominant throughout but that he noticed a relationship between the numerators and denominators of the equivalent fractions that we argue is intuitive multiplicative reasoning: "They are a bit the same, they are in the same times table".

The case study supports the findings of Reimer & Moyer (2005), Suh & Heo (2005) and Steen *et al.* (2006), demonstrating how the use of a virtual manipulative (in this case a rectangle representation with a partitioning tool) provided immediate feedback that allowed George to self-regulate his thinking and make amendments, undertaking a task that would be difficult if not impossible with physical manipulatives. Although paper-based or other tasks with tangible objects provide different affordances, the potential of Fractions Lab includes its dynamic nature and particularly the direct connection between graphical and symbolic representation. Furthermore, it demonstrates how George's misconception is so strongly held that he tries out another statement (2/4=1/8) because he thinks the computer is wrong. However, when the feedback is not what he expects, his assumptions begin to be challenged, and turbulence in his thinking can be observed.

The students' written comments explaining to a friend how the partition tool works supports the findings of Reimer & Moyer (2005) and Suh & Heo (2005) as the majority of students (78%) explicitly commented where a link between the graphical representation and its symbol could be observed.

Despite the open-ended nature of the question, we were surprised when 25% of the students stated that using the representations helped them to understand equivalent fractions. While it is feasible that most students may not have used the number line or liquid measures for an equivalent fractions task, it is also worth exploring further if the way the rectangle is partitioned differently to the other representations also has an impact.

CONCLUSION

We have emerging data that students' interaction with Fractions Lab, and in particular the partitioning tool, provokes them to think conceptually about equivalent fractions. Some appear to be able to capitalise on their intuition, and sometimes to challenge it, discouraging them from simply calculating an answer procedurally. Developing virtual manipulatives that enable students to witness what happens dynamically as they create and partition a fraction appears to have the potential to enhance their conceptual understanding of fraction equivalence by challenging their whole-number bias.

The next step in our research is to evaluate the newly-introduced sets representation to Fractions Lab and to systematically evaluate the effect partitioning using all the representations has on students' conceptual understanding of fraction equivalence. We are also interested in how Fractions Lab may further support students to bridge the gap between their additive reasoning and their multiplicative reasoning.

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