Understanding fractions: interpretations and representations

An iTalk2Learn Guide for teachers

1 Developing a coherent system for fractions learning

Did you know that children's performance in fractions predicts their mathematics achievement in secondary school, above and beyond the contributions of whole number arithmetic knowledge, verbal and non-verbal IQ, working memory, and family education and income?

Seigler et al (2012)

The iTalk2Learn project aims at helping primary school children develop robust knowledge in the field of fractions. Fractions are one of the most difficult aspect of mathematics to teach and learn (Charalambous & Pitta-Pantazi, 2007). The difficulty arises because of the complexity of fractions, such as the number of ways they can be interpreted and the number of representations teachers can draw upon to teach. In this paper we discuss these two aspect of fractions and present the *iTalk2Learn Fractions Interpretations / Representations Matrix* that you may find helpful in your fractions planning and teaching.

1.1 Interpretations of fractions

When teaching fractions, we need to take into account that fractions can be interpreted in several different ways (Kieran, 1976, 1993). The interpretations are part-whole, ratio, operator, quotient, and measure. There is inevitable overlapping between the interpretations, but in Table 1 each interpretation is exemplified using the fraction ¾.

Table 1. Interpretations of fractions, exemplified using 3/4.

Interpretation	Commentary
Part-whole	In part-whole cases, a continuous quantity or a set of discrete objects is partitioned into a number of equal-sized parts. In this interpretation, the numerator must be smaller than the denominator. E.g. a pie is divided into four equal parts (quarters) and three are eaten, so 3/4 of the pie has been eaten. Children need to understand that (a) the parts into which the whole is partitioned must be of equal size; (b) the parts, taken together, must equal to the whole; (c) the more parts the whole is divided into, the smaller the parts become; and (d) the relationship between the parts and the whole is conserved, regardless of the size, shape or orientation of the equivalent parts (Leung, 2009). Part-whole is the most common interpretation used in elementary school exercise books (Alajmi, 2012). It is the interpretation that children perform consistently higher in compared to the other interpretations (Charalambous & Pitta-Pantazi, 2007; Hannula, 2003; Ni, 2001).

Ratio	A fraction can be seen as a ratio of two quantities; in this case it is seen as a part-part interpretation and it is considered to be a comparative index rather than a number (Carraher, 1996). E.g. Three parts out of every four are red. For this interpretation, children need to understand the relative nature of the quantities. They also need to know that when two quantities in the ratio are multiplied by the same positive number, the value of the ratio is unchanged (Leung, 2009).
Operator	A fraction is interpreted as an operator when it is applied as a function to a number, set or object. E.g. showing 3/4 of a pie chart or finding 3/4 of 24. In the operator interpretation, children need to understand that it is possible to interpret a fractional multiplier in a variety of ways; name a single fraction to describe a composite operation, when two multiplicative operations are performed; and relate outputs to inputs (Leung, 2009).
Quotient	The quotient interpretation is the result of a division. It results in a number that can be placed on a number line. E.g. 3 ÷ 4 = 3/4. For this interpretation, children should be able to identify fractions with division and understand the role of the dividend and the divisor in this operation (Leung, 2009).
Measure	In a measure interpretation a unit fraction is identified (e.g. 1/4) and how many units are used repeatedly determine a distance from a predetermined starting point (e.g. 3 x 1/4 = 3/4). In this interpretation, children should be able to use given unit interval to measure any distance from the origin; locate a number on a number line; and identify a number represented by a point on the number line (Leung, 2009) or region (Pantziara & Philippou, 2012).

The part-whole interpretation is the interpretation that most maths curricular around the world focus upon the most (Behr, Lesh, Post, & Silver, 1983; Charalambous, Delaney, Hsu, & Mesa, 2010; Panaoura, Gagatsis, Deliyianni, & Elia, 2009). Which interpretation(s) do you currently include in your planning? Are there any interpretations you have omitted that you could consider including?

1.2 Representations of fractions

What resources do you use in your teaching to model fractions with children?

There are a number of ways that fractions can be represented and several studies demonstrate how the links between representations of fractions and the underlying fractions concepts (Kong, 2008; Paik, 2005; Pitta-Pantazi, Gray, & Christou, 2004; Yang & Reys, 2001) support children's fraction understanding.

The three most commonly cited include representations of area/region, number line and sets of objects (Duval, 2006). Another representation is modelled by liquid measures. Although research into the use of liquid measures is scarce, it has shown potential particularly to conceptual knowledge (Silver, 1983). Table 2 on the next page provides an overview of five types of fractions representations.

Table 2. Representations of fractions.

Representation	**Commentary**
Area, region	The area representation is the most common in textbooks and other instructional materials around the world (Alajmi, 2012; Pantziara & Philippou, 2012). It often uses a figure such as a circle or rectangle with a fractional part shaded. In context-based problems, items such as pies, pizzas and cakes are frequently used.
Number line	The number line representation involves students placing fractions on a number line or identifying the fraction that is shown on a number line. This requires the student to associate a point on the number line with the fraction a/b where each unit segment has been separated into *bc* equivalent line segments.
Sets of objects	Objects are grouped and used to represent whole sets. In the UK, the fractional part is often identified as a different colour or as a subset of the whole. Objects can be as varied as toys, circles, sweets, sticks or children.
Liquid measures	Reference is made to liquid measures in elementary educational materials from the USA (CCSSI, 2010; NCTM, 2000) and in the UK it is commonplace to find young children sharing objects, using everyday language to talk about capacity and solving problems (DfE, 2013; STA, 2012). It is uncommon to consider fractions in this way in Germany, one of the countries where the ELE is being tested.
Symbolic	The symbolic representation of a fraction uses the notation: $\frac{\text{numerator}}{\text{denominator}}$ Additionally, the numerator and denominator take on different meanings according to the type of representation. For example, as an area the denominator represents the number of parts the whole has been cut into and the numerator is the number of parts taken (Mamede, Nunes, & Bryant, 2005), whereas the numerator is compared with the denominator in the ratio model. Furthermore, in the quotient representation the numerator is the number of items to be shared and the denominator is how many people the item(s) must be shared by. The line represents a division.

England's National Numeracy Strategy (DfEE, 1999) and subsequent Primary National Strategy (DfES, 2006) encouraged teachers to use a range of representations including area/region representations beyond circles, sets of objects and number lines. Liquid measures are less commonly used.

Do you teach fractions using a range of resources? Are there any of the representations above that you do not include? How might you be able to include them?

Analyses of text books around the world (Alajmi, 2012; Pantziara & Philippou, 2012), found that teachers consistently show a skewed and limited exposure to a range of interpretations and representations. An over-reliance on any one representation limits students' conceptual understanding of fractions: for instance, area/region representations cannot exceed the number of partitions, which might impede children's learning of improper fractions (Smith, 2002). Indeed, each representation brings its own limitations. The number line, for example, "comprises a difficult model for students to manipulate" (Charalambous & Pitta-Pantazi, 2007) and placing fractions on the number line requires children to know that for the same denominator, the larger the numerator the larger the fraction and for the same numerator, the larger the denominator the smaller the fraction (Nunes, 2004).

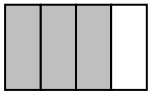
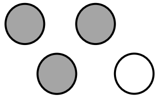
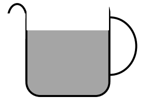
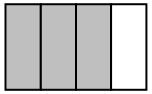
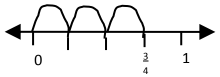
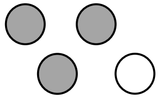
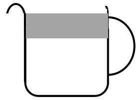
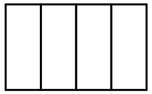
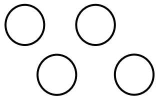
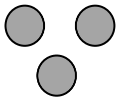
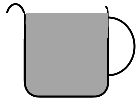
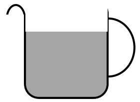
In light of these findings, to ensure robust learning the iTalk2Learn project has set out to enable children access to a range of interpretations and representations within the learning platform because it is well-documented that use of multiple representations improves conceptual learning (Lamon, 1999; Rau, Aleven, & Rummel, 2013), as the constraints of each representation may be mitigated by exposure to others.

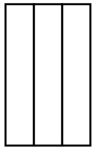
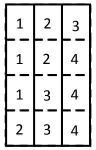
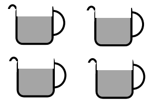
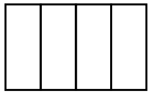
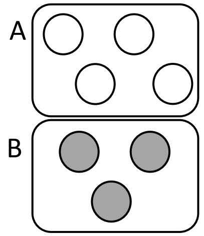
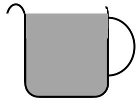
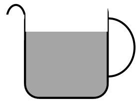
1.3 Bringing together interpretations and representations: The fractions interpretations/representations matrix

Students tend to receive a limited number of interpretations and representations in their curriculum diets with part-whole and area the most common interpretation and representation respectively (Baturo, 2004). The development of each fraction interpretation in isolation does not necessarily lead to understanding of the other concepts (Brousseau et al., 2004, Charalambous & Pitta-Pantazi, 2007) and to have a complete understanding of fractions requires an understanding of the different representations and how they interrelate (Kieren, 1979). Therefore in iTalk2Learn we are utilising a wide range of interpretations and representations in a coherent system for learning fractions to provide a broader diet to learners.

Table 3 presents the iTalk2Learn fractions interpretations/representations matrix that brings together interpretations and representations. It offers examples from each type of fractions context you may use in your teaching. We hope that this will help your fractions planning and teaching.

Table 3: The interpretations and representations matrix

	Symbolic	**Area/region**	**Number line**	**Sets of objects**	**Liquid measures**
Part-whole	3/4	 3/4 of the area is shaded	 3/4 of the number line is grey	 3/4 of the objects are shaded	 3/4 of the jug is filled
Ratio	3/4	 3 out of 4 parts are shaded	 3 out of 4 parts have jumped along the number line (3 x 1/4)	 3 out of 4 objects are shaded	 3 out of 4 parts of the liquid is water (the rest is oil)
Operator	3/4	 Finding 3/4 of the region gives: 	 Finding 3/4 of the line segment gives: 	 Finding 3/4 of the objects gives: 	 Finding 3/4 of the liquid gives: 

Quotient	3/4	 ← 3 are shared by 4 →  So each person receives 3/4 each	Road relay String	 3 objects are shared by 4, so each person receives 1/4 of each object = 3/4 each	 3 jugs are shared by 4, so each new jug receives 1/4 of each = 3/4 jug 
Measure	3/4	 The shaded object is 3/4 the white object 	 The second line segment is 3/4 the first 	 Set B is 3/4 of Set A	  The second jug is 3/4 the first